1 a) $\ell(\theta_0) \cong \ell(\hat\theta) + (\theta_0 - \hat\theta)\,\dot\ell(\hat\theta) + \frac{1}{2}(\theta_0 - \hat\theta)^2\,\ddot\ell(\hat\theta)$

$\qquad\;\; = \ell(\hat\theta) + \quad 0 \quad + \frac{1}{2}(\theta_0 - \hat\theta)^2\,\ddot\ell(\hat\theta)$

b) $$\frac{1}{n}\frac{d^2\ell}{d\theta^2}(\theta_0) = \frac{1}{n}\sum_{i=1}^{n}\frac{d^2\ell_{x_i}}{d\theta^2}(\theta_0)$$

$$\overset{LLN}{=} E\,\frac{d^2\ell_{x_i}}{d\theta^2}(\theta_0)$$

$$= E\,\frac{d}{d\theta^2}\log f_\theta(X_1)\Big|_{\theta=\theta_0} = -I(\theta_0)$$

c) $$LRT = \frac{L(\theta_0)}{\max_\theta L(\theta)} = \frac{L(\theta_0)}{L(\hat\theta)}$$

$$-2\log LRT = -2\log\frac{L(\theta_0)}{L(\hat\theta)}$$

$$= -2\left(\ell(\theta_0) - \ell(\hat\theta)\right)$$

$$\cong -2\left(\ell(\hat\theta) + \frac{1}{2}(\theta_0 - \hat\theta)^2\,\ddot\ell(\hat\theta) - \ell(\hat\theta)\right)$$

$$= -(\theta_0 - \hat\theta)^2\,\ddot\ell(\hat\theta)$$

d) We know

$$\sqrt{n}\,(\hat\theta - \theta_0) \longrightarrow N(0, I(\theta_0)^{-1})$$

$$\frac{1}{n}\ddot\ell(\theta_0) \longrightarrow -I(\theta_0)$$

So $$(\hat\theta - \theta_0)^2\,\ddot\ell(\theta_0) = \left[\sqrt{n}\,(\hat\theta - \theta_0)\right]^2\frac{1}{n}\ddot\ell(\theta_0)$$

$$= \Big[\underbrace{\sqrt{n}\,(\theta - \theta_0)\cdot\sqrt{\tfrac{1}{n}\ddot\ell(\theta_0)}}_{\to N(0,1)}\Big]^2$$

$$\longrightarrow \chi^2_1$$

2. $X_1 \dots X_n \sim Un(0, \theta)$

a) $EX_i = \frac{1}{2}\theta$

MoM of θ: $\widehat{EX_i} = \frac{1}{n}\Sigma X_i$

$$\widehat{\tfrac{1}{2}\theta} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$\hat{\theta}_{MoM} = \frac{2}{n}\sum_{i=1}^{n} X_i$$

b) Clearly $\hat{\theta}_{MoM}$ is unbiased

moreover, as $V(\hat{\theta}) = \frac{4}{n} V(X_1) \longrightarrow 0$

we have that $\hat{\theta}_{MoM}$ is consistent.

3)

a) $\hat{\lambda} = \dfrac{n}{\sum_{i=1}^{n} x_i Y_i}$

$\boxed{\text{10 MARKS}}$

b) Consider joint density.

$\boxed{\text{5 MARKS}}$

$$f_{\theta, y}(y) = \lambda^n \prod_{i=1}^{n} x_i \, e^{-\lambda \sum_{i=1}^{n} x_i y_i}$$

$$= \prod_{i=1}^{n} x_i \times \underbrace{\lambda^n e^{-\lambda \frac{n}{\hat{\lambda}}}}_{\text{So } h(\lambda, \hat{\lambda})}$$

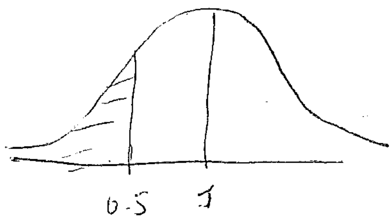
So by the factorization theorem, $\hat{\lambda}$ is sufficient

C i) Consider $\dfrac{d^2 l}{d\lambda^2} = -\dfrac{n}{\lambda^2}$

so $\hat{\lambda} \sim N\left(\lambda, \dfrac{\lambda^2}{n}\right)$ for large n

$H_0 : \lambda = 1$

$H_1 \quad \lambda \neq 1$



$$p\text{-value} = 2 \cdot P_{H_0}(\hat{\lambda} \leq 0.5)$$

$$= 2 \cdot P_{H_0}\left(\frac{\hat{\lambda} - 1}{\sqrt{1/40}} \leq \frac{0.5 - 1}{\sqrt{1/40}} \right)$$

$$= 2\, P(Z \leq -3.16)$$

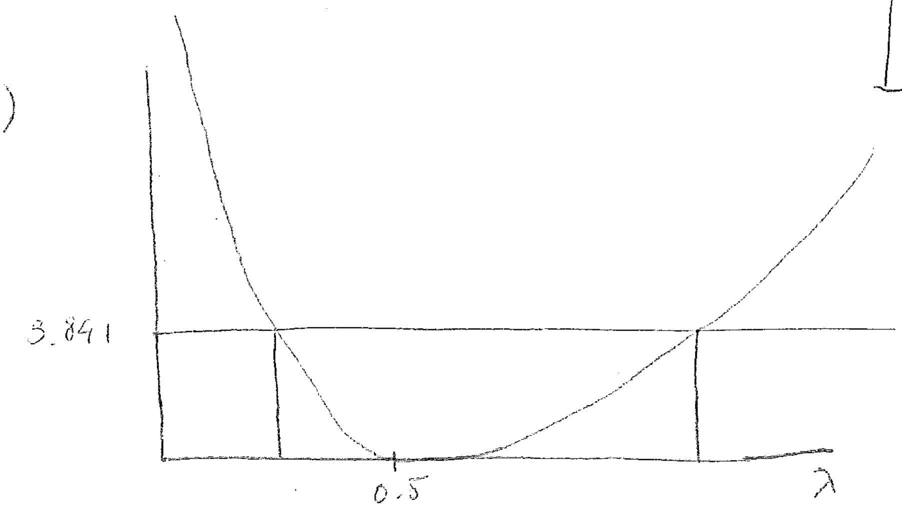
$$= 0.0016 < 0.05$$

$\boxed{\text{10 MARKS}}$

So reject H_0

10 MARKS

c (ii)



We use $\hat{\lambda} = \frac{40}{80} = 0.5$

$$\Lambda = -2 \log \frac{L_Y(\theta)}{L_Y(\hat{\theta})} \Big| \theta \sim \chi^2_1 \text{ asymptotically}$$

note $\Lambda = -2(\ell_Y(\theta) - \ell_Y(\hat{\theta}))$

$$= -2 \Big\{ (n \log \lambda + \sum_{i=1}^{n} \log x_i - \lambda \sum x_i Y_i)$$

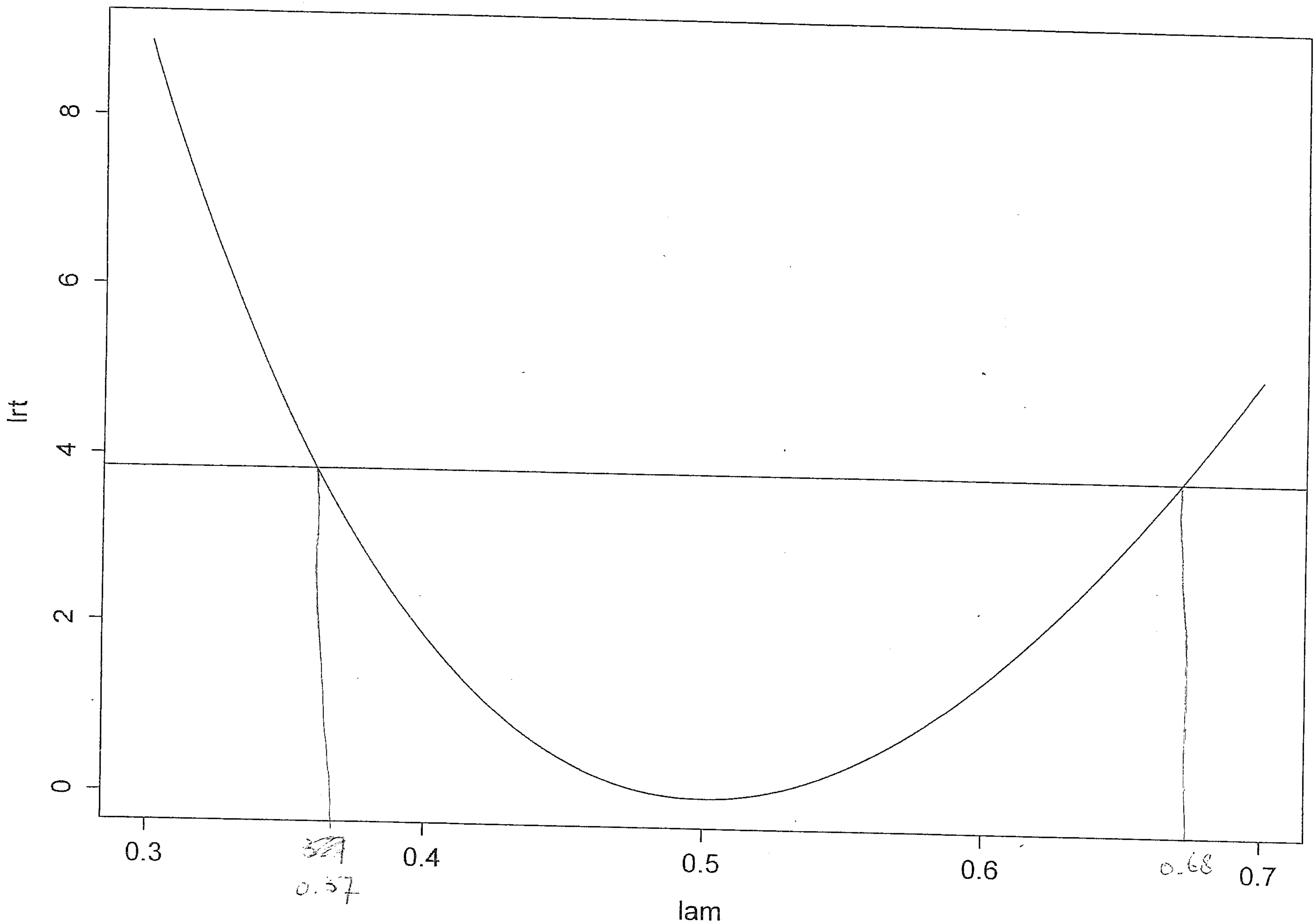
$$- (n \log \hat{\lambda} + \sum \log x_i - \hat{\lambda} \sum x_i Y_i) \Big\}$$

$$= -2n \log \frac{\lambda}{\hat{\lambda}} + (\lambda - \hat{\lambda}) \sum_{i=1}^{n} x_i Y_i \times 2$$

So $\Lambda_{observed} = 80 \log \frac{0.5}{\lambda} + (\lambda - 0.5) \cdot 80 \cdot 2$

Given the plot we find,

$(0.37, 0.68)$


lrt
0
2
4
6
8
lam
0.3
0.4
0.5
0.6
0.7
0.37
0.68

RESIT
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$H_0: \mu = 1$
$H_1: \mu = 2$

4 a) $CR_\alpha = \left\{ x \;\middle|\; \frac{L_0(x)}{L_1(x)} \leq c_\alpha \right\}$

where $L_1(x) = \frac{1}{2} e^{-x/2}$
$L_0(x) = e^{-x}$

$$\alpha = 0.05 = P(X \in CR_{0.05} \mid H_0)$$

$$= P_{H_0}\left(2 e^{+X_1/2 - X_1} \leq c_\alpha \right)$$

$$= P_{H_0}\left(-X/2 \leq c_\alpha^{(2)} \right)$$

$$= P_{H_0}\left(X \geq c_\alpha^{(3)} \right)$$

$$= e^{-c_\alpha^{(3)}}$$

So $c_\alpha^{(3)} = -\log 0.05$

$$= 2.996$$

<u>So</u> $CR_{0.05} = \{ x \geq 2.996 \}$

b) Power $= P_{H_1}(CR_{0.05})$

$$= P_{H_1}(X \geq 2.996)$$

$$= e^{-2.996/2} =$$

$$= 0.224$$